

PRACTICE WORKSHEET + FULL ANSWER KEY

Differentiation and integration

Senior calculus. Work through the sections in order, or choose the one that matches this week's topic.

QUESTIONS	SECTIONS	ANSWER KEY	WORKING SPACE
105	10	At the back	Ruled

Name _____ Class _____ Date _____

Before you start

A derivative measures instantaneous rate of change. An integral accumulates signed change. Keep units and the interval visible so the result has meaning in context.

WORKED EXAMPLE

Find the stationary point of $f(x) = x^2 - 6x + 8$.

- 1 Differentiate: $f'(x) = 2x - 6$.
- 2 Set $f'(x) = 0$, so $x = 3$.
- 3 $f(3) = -1$. The parabola opens upwards, so $(3, -1)$ is a minimum.

Use this method as your model. Set out your own working the same way.

Common mistakes to avoid

- Forgetting the constant of integration
- Assuming every stationary point is a maximum

There are 105 questions across 10 sections. Write in the ruled space, then mark your work against the answer key at the back. Topics covered: Derivatives, Stationary points, Definite integrals.

SECTION A Rates of change and first derivatives

A derivative measures instantaneous rate of change. Keep the variable and its units in view.

1 Find the limit of $(x^2 - 4)/(x - 2)$ as x approaches 2.

3 Find the average rate of change of $f(x) = x^2$ between $x = 1$ and $x = 3$.

2 Differentiate $y = x^3$ from first principles and state the result.

4 Explain the difference between an average rate of change and an instantaneous rate.

5

Differentiate $y = 7$.

6

Differentiate $y = 5x$.

7

Differentiate $y = x^4$.

8

Differentiate $y = 3x^4 - 2x + 7$.

9

Find $f'(2)$ for $f(x) = x^3$.

10

Differentiate $y = 1/x$ by writing it as a power first.

SECTION B Differentiation rules*Name the rule you are using before you apply it.*

11

Differentiate $y = (2x + 1)^3$.

12

Differentiate $y = x^2 e^x$.

13

Differentiate $y = \sin(x)$.

14

Differentiate $y = \cos(3x)$.

15

Differentiate $y = \ln(x)$.

16

Differentiate $y = e^{2x}$.

17

Differentiate $y = x/(x + 1)$.

18

Differentiate $y = \sqrt{x}$.

19

Differentiate $y = (x^2 + 1)^5$.

20

Differentiate $y = 4x^3 - 6x^2 + 2x - 9$.

21

Find the second derivative of $y = x^4 - 3x^2$.

22

Differentiate $y = 3 \sin(x) - 2 \cos(x)$.

SECTION C Tangents, stationary points and motion

Set the derivative to zero for stationary points, then use a sign test or the second derivative to classify them.

23 Find the gradient of $y = x^3$ at $x = 2$.

24 Find the equation of the tangent to $y = x^2$ at $x = 3$.

25 Find the stationary point of $f(x) = x^2 - 6x + 8$ and classify it.

26 Find the stationary points of $y = x^3 - 3x$ and classify them.

27 Classify the stationary point of $y = 4 - x^2$.

28 Find where $y = x^3 - 3x^2$ is increasing.

29 Find the point of inflection of $y = x^3 - 3x^2$.

30 A position is $s(t) = t^2 + 3t$ metres. Find the velocity at $t = 2$ seconds.

31 For $s(t) = t^3 - 6t^2 + 9t$, find when the object is at rest.

32 For that motion, find the acceleration at $t = 4$.

33 A balloon's radius grows at 2 cm/s. Find dV/dt when $r = 5$ cm, using $V = (4/3)\pi r^3$.

SECTION D Integration

Integration reverses differentiation. Add a constant for an indefinite integral; use limits for a definite one.

34 Integrate $6x^2$ with respect to x .

35 Integrate $4x^3 - 2$ with respect to x .

36 Evaluate the integral of $2x$ from $x = 1$ to $x = 3$.

37 Integrate $1/x$ with respect to x , for $x > 0$.

38 Integrate e^{3x} with respect to x .

39 Integrate $\sin(x)$ with respect to x .

40

Evaluate the integral of x^2 from $x = 0$ to $x = 3$.

41

Find the area between $y = x^2$ and the x -axis from $x = 0$ to $x = 2$.

42

Evaluate the integral of $(2x + 1)$ from $x = 0$ to $x = 4$.

43

Integrate $(3x + 2)^4$ with respect to x .

44

A velocity is $v(t) = 6t$. Find the displacement from $t = 0$ to $t = 5$.

45

Explain why an indefinite integral needs a constant of integration.

SECTION E Modelling with calculus

State the model, the variable and the units before you calculate anything.

46

A cost function is $C(x) = 0.5x^2 + 20x + 300$. Find the marginal cost at $x = 40$.

47

For that cost function, find the output that minimises the average cost $C(x)/x$.

48

A rectangle has perimeter 60 m. Use calculus to find the maximum area.

49

A tank drains so that $V(t) = 400 - 8t$ litres. Find the rate of change and explain its sign.

50

A temperature follows $T(t) = 20 + 15e^{-0.2t}$. Find dT/dt at $t = 0$.

51

Explain why a stationary point is not always a maximum or a minimum.

52

A student integrates $3x^2$ and writes x^3 . Explain what is missing.

53

Explain what the definite integral of a velocity function represents.

54

A box has a square base of side x and volume 32 cm^3 . Write its surface area in terms of x .

55

Explain why a graph is useful before optimising a function.

SECTION F Further differentiation

State the rule, then the derivative of each part, then combine.

56

Differentiate $y = x^5 - 4x^3 + x$.

57

Differentiate $y = 3\sqrt{x}$.

58

Differentiate $y = x^2 \sin(x)$.

59

Differentiate $y = \ln(3x)$.

60

Differentiate $y = e^{x^2}$.

61

Differentiate $y = (x + 1)/(x - 1)$.

62

Differentiate $y = \cos^2(x)$.

63

Differentiate $y = 1/(2x + 3)$.

64

Differentiate $y = x \ln(x)$.

65

Differentiate $y = \sqrt{4x + 1}$.

SECTION G Related rates and approximation

Write the relationship first, then differentiate both sides with respect to time.

66

A square's side grows at 3 cm/s. Find dA/dt when the side is 10 cm.

67

A circle's radius grows at 0.5 cm/s. Find dA/dt when $r = 4$ cm.

68

A cube's edge grows at 2 mm/s. Find dV/dt when the edge is 5 mm.

69

A 5 m ladder slides down a wall. Explain how to relate the rates of the base and the top.

70

Estimate the change in $y = x^2$ when x changes from 3 to 3.01.

71

State what $dy/dx = 0$ tells you about a curve.

72

A cone has $V = (1/3) \pi r^2 h$. With h fixed, find dV/dr .

73

Water fills a tank at 5 L/min. Write this as a derivative with its units.

74

A balloon's volume grows at $12 \text{ cm}^3/\text{s}$. Find dr/dt when $r = 2 \text{ cm}$, using $V = (4/3) \pi r^3$.

75

Explain the chain rule in one sentence about linked rates.

SECTION H Areas, volumes and further integration

Sketch the region first so you know which curve is on top and where the limits are.

76

Find the area between $y = x^2$ and $y = x$ from $x = 0$ to $x = 1$.

77

Find the area under $y = 1/x$ from $x = 1$ to $x = e$.

78

Evaluate the integral of $\cos(x)$ from $x = 0$ to $x = \pi/2$.

79

Integrate $5e^{2x}$ with respect to x .

80

Integrate $1/(2x + 1)$ with respect to x .

81

Find the volume when $y = x$ is rotated about the x -axis from $x = 0$ to $x = 3$.

82

Evaluate the integral of $(x^2 + 2x)$ from $x = 1$ to $x = 2$.

83

Integrate $\sec^2(x)$ with respect to x .

84

Use integration by parts to integrate $x e^x$.

85

Explain why an area below the x -axis gives a negative definite integral.

SECTION I Motion and differential equations

Differentiate to go from position to velocity to acceleration; integrate to go back.

86 For $s(t) = 2t^3 - 9t^2 + 12t$, find the velocity function.

87 For that motion, find when the object is momentarily at rest.

88 For that motion, find the acceleration at $t = 3$.

89 A particle has $v(t) = 4 - 2t$. Find its displacement from $t = 0$ to $t = 4$.

90 For that particle, find the total distance travelled from $t = 0$ to $t = 4$.

91 Solve $dy/dx = 3x^2$ given that $y = 5$ when $x = 0$.

92 Solve $dy/dx = 2y$ by separating the variables.

93 A population satisfies $dP/dt = 0.03P$. Describe the model.

94 An object cools so that $dT/dt = -k(T - 20)$. Explain the 20.

95 Explain the difference between displacement and distance as integrals.

SECTION J Mixed review and challenge

Check whether the question wants a value, a point, an equation or an interpretation.

96 Differentiate $y = (3x - 2)^5$.

97 Find the stationary points of $y = x^4 - 8x^2$.

98 Classify the stationary point at $x = 0$ for that curve.

99 Find the equation of the normal to $y = x^2$ at $x = 1$.

100 Evaluate the integral of $(6x^2 - 4x)$ from $x = 0$ to $x = 2$.

101 Find the maximum value of $y = 12x - x^2$.

102 An open-topped box has a square base of side x and volume 500 cm^3 . Write its surface area in terms of x .

103

For that box, use calculus to find the x that minimises the surface area.

104

State the derivative of a constant multiple $k f(x)$.

105

Explain why endpoints must be checked when optimising on a closed interval.

Answer key

Work each section before you check it. These are model answers: an equivalent method, or a different well-supported interpretation, can be just as correct.

SECTION A - RATES OF CHANGE AND FIRST DERIVATIVES

1. 4, because the expression simplifies to $x + 2$ for x not equal to 2.
2. $dy/dx = 3x^2$.
3. 4, from $(9 - 1)/(3 - 1)$.
4. The average rate uses two points on the curve; the instantaneous rate is the gradient of the tangent at a single point.
5. $dy/dx = 0$.
6. $dy/dx = 5$.
7. $dy/dx = 4x^3$.
8. $dy/dx = 12x^3 - 2$.
9. 12.
10. $dy/dx = -1/x^2$.

SECTION B - DIFFERENTIATION RULES

11. $dy/dx = 6(2x + 1)^2$, by the chain rule.
12. $dy/dx = e^x(x^2 + 2x)$, by the product rule.
13. $dy/dx = \cos(x)$.
14. $dy/dx = -3 \sin(3x)$.
15. $dy/dx = 1/x$.
16. $dy/dx = 2e^{2x}$.
17. $dy/dx = 1/(x + 1)^2$, by the quotient rule.
18. $dy/dx = 1/(2\sqrt{x})$.
19. $dy/dx = 10x(x^2 + 1)^4$.
20. $dy/dx = 12x^2 - 12x + 2$.
21. $d^2y/dx^2 = 12x^2 - 6$.
22. $dy/dx = 3 \cos(x) + 2 \sin(x)$.

SECTION C - TANGENTS, STATIONARY POINTS AND MOTION

23. 12.
24. $y = 6x - 9$.
25. (3, -1), a minimum, since the parabola opens upwards.
26. (1, -2) is a minimum and (-1, 2) is a maximum.
27. (0, 4) is a maximum.
28. For $x < 0$ and $x > 2$, because $dy/dx = 3x^2 - 6x$ is positive there.
29. (1, -2), where the second derivative changes sign.
30. $v(t) = 2t + 3$, so $v(2) = 7$ m/s.
31. At $t = 1$ s and $t = 3$ s, where $v = 3t^2 - 12t + 9 = 0$.
32. $a(t) = 6t - 12$, so $a(4) = 12$ m/s².
33. $dV/dt = 4\pi r^2 (dr/dt) = 200\pi$ cm³/s.

SECTION D - INTEGRATION

34. $2x^3 + C$.
35. $x^4 - 2x + C$.
36. 8, because $[x^2]$ gives $9 - 1$.
37. $\ln(x) + C$.
38. $(1/3)e^{3x} + C$.
39. $-\cos(x) + C$.
40. 9.
41. $8/3$ square units.
42. 20.
43. $(3x + 2)^5/15 + C$.
44. 75 m.
45. Any constant differentiates to zero, so infinitely many functions share the same derivative.

SECTION E - MODELLING WITH CALCULUS

46. $C'(x) = x + 20$, so the marginal cost is \$60 per unit.
47. Average cost is $0.5x + 20 + 300/x$, least when $x^2 = 600$, about 24.5 units.
48. $A = x(30 - x)$ and $A' = 30 - 2x = 0$ gives $x = 15$ m, so the area is 225 m².
49. $dV/dt = -8$ L/min; the negative sign shows the volume is falling.
50. -3 degrees per minute.
51. The gradient can also be zero at a point of inflection, where the curve does not turn.
52. The constant of integration; the answer is $x^3 + C$.
53. The displacement over the interval, including direction, rather than the total distance travelled.
54. $A = x^2 + 128/x$, using height $h = 32/x^2$ for an open-topped box.
55. It shows how many turning points exist and whether a stationary value is a local or a global optimum.

SECTION F - FURTHER DIFFERENTIATION

56. $dy/dx = 5x^4 - 12x^2 + 1$.
57. $dy/dx = 3/(2\sqrt{x})$.
58. $dy/dx = 2x \sin(x) + x^2 \cos(x)$.
59. $dy/dx = 1/x$.
60. $dy/dx = 2x e^{x^2}$.
61. $dy/dx = -2/(x - 1)^2$.
62. $dy/dx = -2 \cos(x) \sin(x)$.
63. $dy/dx = -2/(2x + 3)^2$.
64. $dy/dx = \ln(x) + 1$.
65. $dy/dx = 2/\sqrt{4x + 1}$.

SECTION G - RELATED RATES AND APPROXIMATION

66. $60 \text{ cm}^2/\text{s}$, from $dA/dt = 2s(ds/dt)$.
67. $4\pi \text{ cm}^2/\text{s}$.
68. $150 \text{ mm}^3/\text{s}$.
69. Differentiate $x^2 + y^2 = 25$ with respect to time: $2x(dx/dt) + 2y(dy/dt) = 0$.
70. About 0.06, using $dy = 2x dx = 6(0.01)$.
71. The tangent is horizontal at that point.
72. $dV/dr = (2/3)\pi r h$.
73. $dV/dt = 5 \text{ L/min}$.
74. $dr/dt = 3/(4\pi) \text{ cm/s}$, from $12 = 4\pi r^2 (dr/dt)$.
75. The rate of change of a composite quantity is the product of the rates of change of the linked parts.

SECTION H - AREAS, VOLUMES AND FURTHER INTEGRATION

76. $1/6$ square units.
77. 1 square unit.
78. 1.
79. $(5/2)e^{2x} + C$.
80. $(1/2)\ln|2x + 1| + C$.
81. 9π cubic units.
82. $16/3$.
83. $\tan(x) + C$.
84. $x e^x - e^x + C$.
85. The signed height is negative there, so the accumulated value is negative; take the modulus for a physical area.

SECTION I - MOTION AND DIFFERENTIAL EQUATIONS

86. $v(t) = 6t^2 - 18t + 12$.
87. At $t = 1 \text{ s}$ and $t = 2 \text{ s}$.
88. $a(t) = 12t - 18$, so $a(3) = 18$.
89. 0, because it travels forward then back by the same amount.
90. 8 units: 4 forward up to $t = 2$, then 4 back.
91. $y = x^3 + 5$.
92. $y = Ae^{2x}$.
93. Exponential growth, with $P = P_0 e^{0.03t}$.
94. It is the surrounding temperature; the object cools towards it and the rate falls as the difference shrinks.
95. Displacement integrates signed velocity; distance integrates the modulus of the velocity.

SECTION J - MIXED REVIEW AND CHALLENGE

96. $dy/dx = 15(3x - 2)^4$.
97. $(0, 0)$, $(2, -16)$ and $(-2, -16)$.
98. A local maximum, because the second derivative $12x^2 - 16$ is negative there.
99. $y = -x/2 + 3/2$.

100. 8.
101. 36, at $x = 6$.
102. $A = x^2 + 2000/x$.
103. $x = 10 \text{ cm}$, from $2x - 2000/x^2 = 0$.
104. $k f'(x)$; the constant is carried through unchanged.
105. The greatest or least value can occur at an endpoint rather than at a stationary point.

Review and next steps

Mark your work against the answer key, record your score for each section, then answer the three questions below. Bring this page to your next lesson so your tutor can start from what actually blocked you.

Which question did you find hardest, and what exactly stopped you?

Rewrite the corrected working for that question here.

Which section do you want to practise again before your next lesson?

Score record

SECTION	QUESTIONS	NUMBER CORRECT	PRACTISE AGAIN?
A. Rates of change and first derivatives	10		
B. Differentiation rules	12		
C. Tangents, stationary points and motion	11		
D. Integration	12		
E. Modelling with calculus	10		
F. Further differentiation	10		
G. Related rates and approximation	10		
H. Areas, volumes and further integration	10		
I. Motion and differential equations	10		
J. Mixed review and challenge	10		
Total	105		

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